Computer Poker Tutorial @ EC 2016, part 2: Equilibrium Computation

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Google DeepMind

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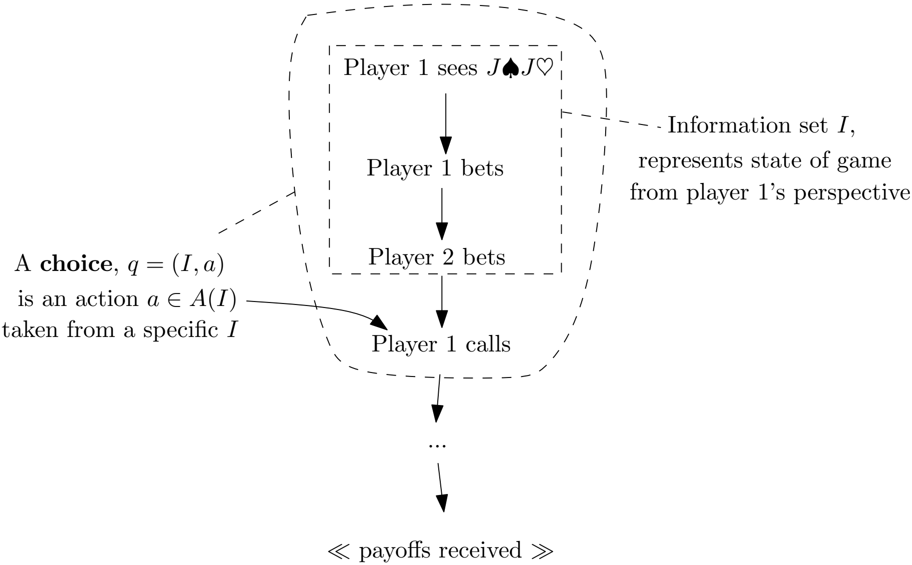
Outline

- Optimization using the sequence form
- Counterfactual Regret (CFR) Minimization
- Selected CFR extensions
- Solving 2-player Heads-up Limit Texas Hold'em
- Two open problems

Slides and references available at:
`http://mlanctot.info/ecpokertutorial2016/`

Extensive-form Games



Information sets: $I \in \mathcal{I}$, (context-specific) choices $q = (I, a) \in Q$.

Defaults

Unless otherwise noted, assume:

- Notation based on [Osborne & Rubinstein '94]
- Perfect recall
- Mixed (randomized) strategies are used σ
- Two players: $N = 2$
  - subscript i refers to a player i
  - subscript $-i$ refers to the opponent(s) of player i
- Zero-sum: for every game outcome z, $\sum_{i=1}^{N} u_i(z) = 0$
  - Set of Nash eq. profiles $\{\sigma^*\} \Leftrightarrow$ set of minimax profiles
  - Expected values for eq. $u_i(\sigma^*)$ are unique
  - Nash strategies for player i are interchangeable: $u_i(\sigma^*_{i,1}, \sigma^*_{-i}) = u_i(\sigma^*_{i,2}, \sigma^*_{-i})$

Sequence-Form Representation

Refs: [Koller, Megiddo, von Stengel '94][von Stengel '07]

Let $Q_i = \{(I, a) \mid I \in \mathcal{I}, a \in A(I)\}$ be the set of choices for player i.

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Encode *realization plan* for player i using constraints; $\Delta Q :=$ all $\mathbf{x}$ such that

- $x_i(q_\emptyset) = 1$
- $x_i(q) = \sum_{q' \in succ_i(q)} x_i(q')$

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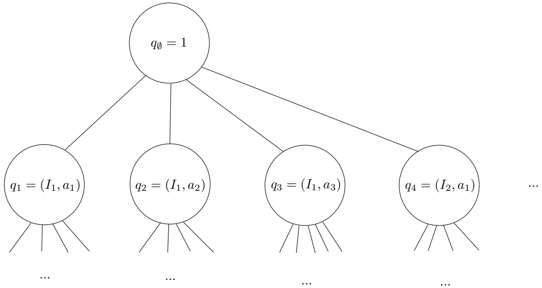

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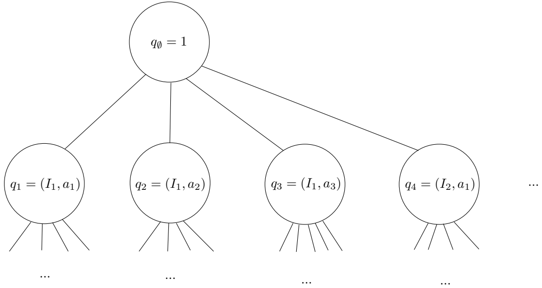

$\Rightarrow$ Much more space-efficient than mixture over pure strategies!

Sequence-Form Linear Programming

For two-player zero-sum, setup an optimization problem:

$$\max_{\mathbf{x}\in\Delta Q_1}\min_{\mathbf{y}\in\Delta Q_2}\mathbf{xAy}=\min_{\mathbf{y}\in\Delta Q_2}\max_{\mathbf{x}\in\Delta Q_1}\mathbf{xAy},$$

Subject to $\mathbf{Ex}=\mathbf{e},\mathbf{x}\geq\mathbf{0},\mathbf{x}\cdot\mathbf{1}=1$.

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Here:

- $\mathbf{A}$ has an entry for every q_1 and q_2 that result in terminal states.
- $\mathbf{E}$ encodes the structure of Q_i
- $\mathbf{e}$ is $(1, 0, 0, 0, ...)^T$

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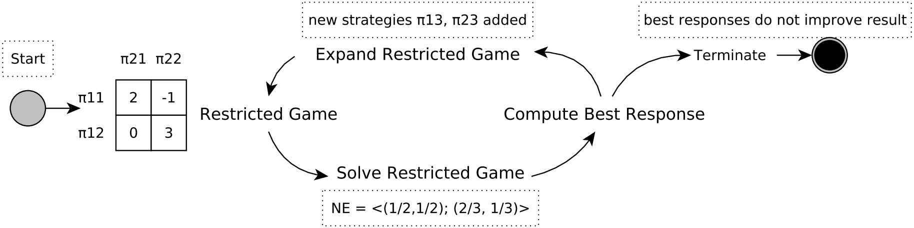
For large games, in practice, $\mathbf{A}$ is sparse and so memory requirements are closer to $|Q_1| + |Q_2|$.

Double-Oracle Methods

Refs: [Zinkevich et al. '06][Bosansky et. al '14]



In extensive-form:

- Each row/column corresponds to a realization plan
- Heuristics to compute "good" best responses
- Could still require enumerating entire space

Nesterov's Excessive Gap Technique (EGT)

Refs: [Gilpin et al. '07][Hoda et al. '10]

Recall:

$$\max_{\mathbf{x}\in\Delta Q_1} f(\mathbf{y}) = \min_{\mathbf{y}\in\Delta Q_2} \phi(\mathbf{x}),$$

where $f(\mathbf{y}) = \min_{\mathbf{y}\in\Delta Q_2} z, \quad \phi(\mathbf{x}) = \max_{\mathbf{x}\in\Delta Q_1} z, \quad z = \mathbf{xAy}.$

Use a strongly convex function d_i on ΔQ_i, and define:

$$f_{\mu_2}(\mathbf{y}) = \min_{\mathbf{y}\in Q_2}\{z + \mu_2 d_2(\mathbf{y})\}$$

$$\phi_{\mu_1}(\mathbf{x}) = \max_{\mathbf{x}\in Q_1}\{z - \mu_1 d_1(\mathbf{x})\}$$

Then $\quad f_{\mu_2}(\mathbf{y}) \geq \phi_{\mu_1}(\mathbf{x}) \quad \Rightarrow \quad 0 \leq \phi(\mathbf{y}) - f(\mathbf{x}) \leq \mu_1 d_1^\top + \mu_2 d_2^\top$, so iteratively compute $(\mathbf{x}^k, \mathbf{y}^k, \mu_1^k, \mu_2^k)$ such that $\mu_i^{k+1} < \mu_i^k$ by gradient descent.

Theorem: EGT computes an ϵ-equlibrium in $O(1/\epsilon)$ iterations.

Counterfactual Regret Minimization (CFR)

CFR [Zinkevich et al. 2008] is iterative strategy-updating algorithm:

	$t = 1$
Player 1 strategies:	σ_1^1
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Let R_i^T be the **external regret** of using σ^t after T steps:

$$R_i^T = \max_{a \in A} \mathbb{E}\left[\sum_{t=1}^{T} \left(u_i(a, \sigma_{-i}^t) - u_i(\sigma_i^t, \sigma_{-i}^t)\right)\right]$$

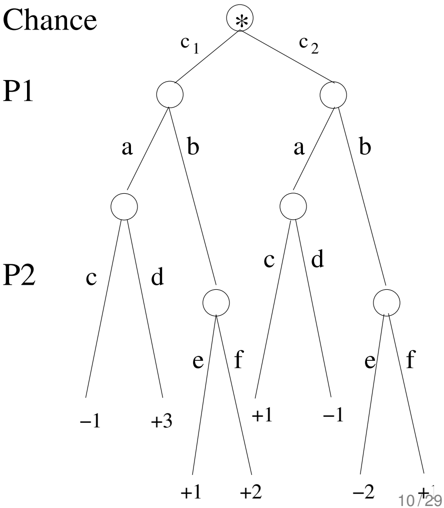
$R_i^T / T \leq \epsilon \quad \Rightarrow \quad$ the average profile $(\bar{\sigma}_1^T, \bar{\sigma}_2^T)$ is a 2ϵ-Nash.

- σ is ϵ-Nash if a player can do ϵ better by switching to σ_i'.
- σ is Nash if no player can do better by switching strategies.

Terminology I : Extensive-Form Games

Example:

An **extensive-form game** is represented in tree form.



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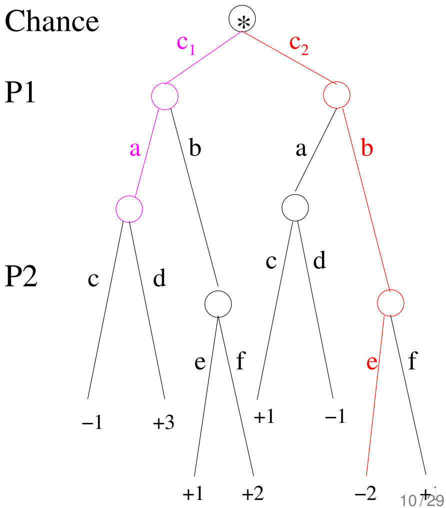

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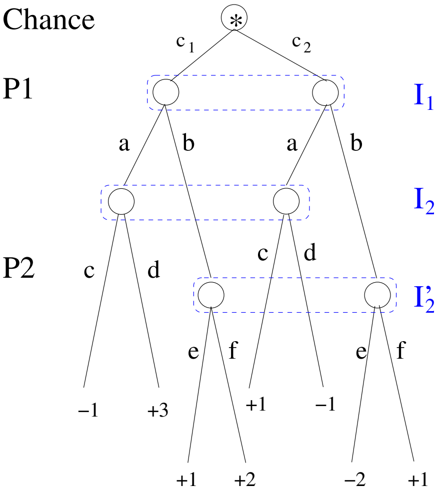

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- An information set $I_i \in \mathcal{I}$ is an information set for player i.

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- $A(I_i)$ is the action set for i at information set I_i.

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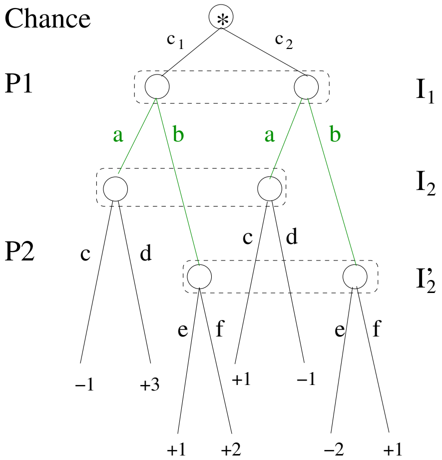
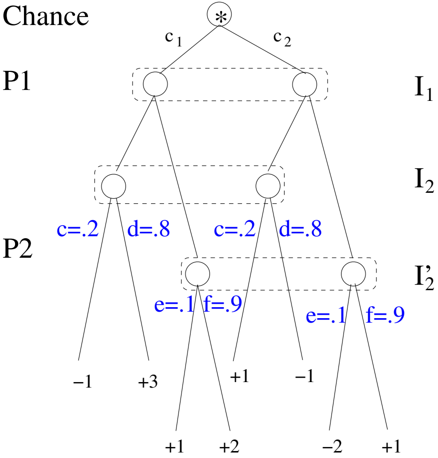


Terminology II : Strategies

A **strategy** $\sigma_i \in \Sigma_i$ is a distribution from $I_i \rightarrow A(I_i)$.



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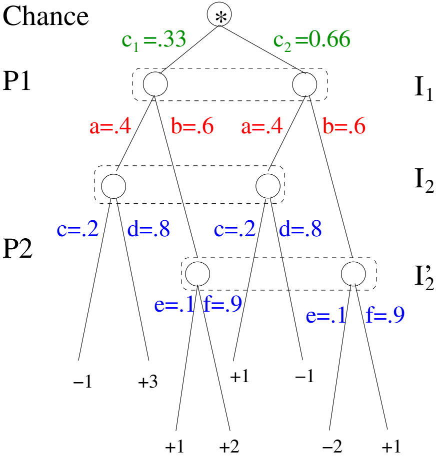

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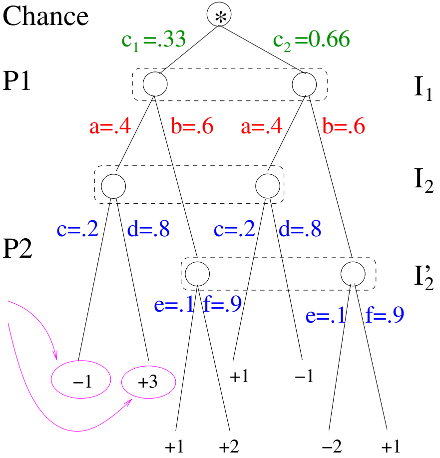

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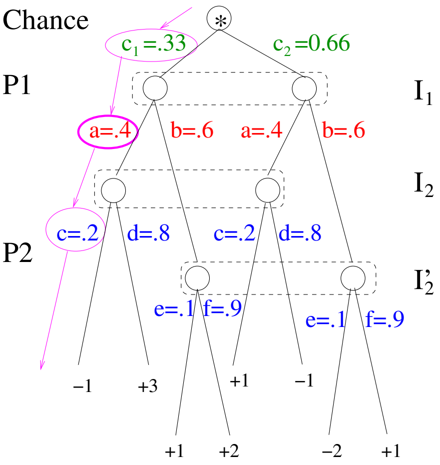
- $\pi^\sigma(h)$ is a product of probabilities along history h. $\pi_i^\sigma(h)$ is player i's contribution.



CFR Algorithm (Overview)

Refs: [Zinkevich et al. '08][Hart & Mas-Colell '00]

1. Minimize **average immediate counterfactual regret** $R^T_{i,\text{imm}}(I)$
2. **Theorem 3**: Overall regret bounded by

$$R^T_i/T \leq \sum_{I \in \mathcal{I}_i} R^{T,+}_{i,\text{imm}}(I)$$

3. **Theorem 4**: Using regret-matching to update strategies, σ^t at each information set, then

$$R^T_i/T \leq \frac{\Delta_{u,i}|\mathcal{I}_i|\sqrt{|A_i|}}{\sqrt{T}}$$

where $\Delta_{u,i}$ is a payoff range for i.

CFR Algorithm (Example)

Define **counterfactual value** as

$$v_i(\sigma, I) = \sum_{h \in I, z \in Z} \pi^{\sigma}_{-i}(h)\pi^{\sigma}(h, z)u_i(z)$$

Define $v_i(\sigma_{(I \to a)}, I)$ similarly, except take a at I

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Repeat until sufficiently small ϵ:

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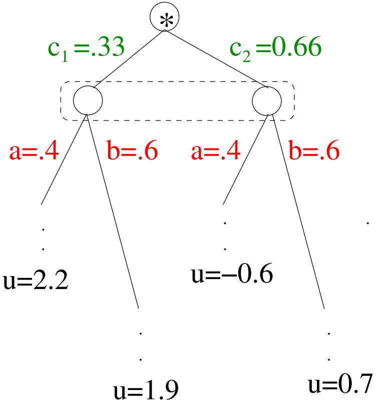

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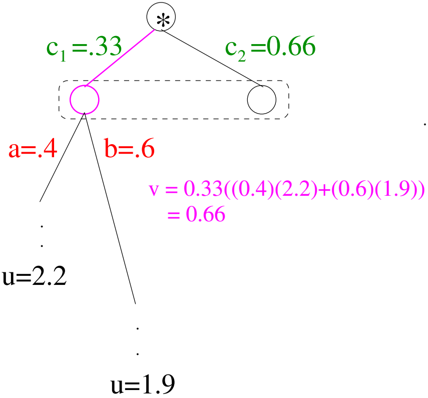

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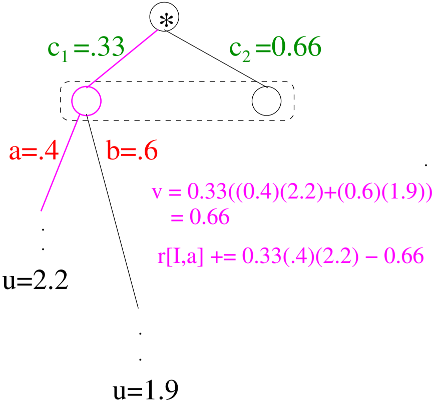

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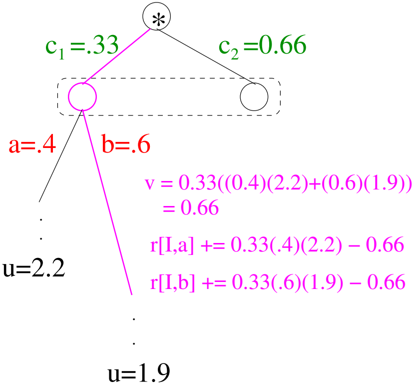

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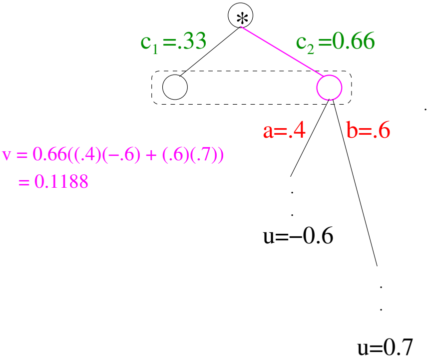

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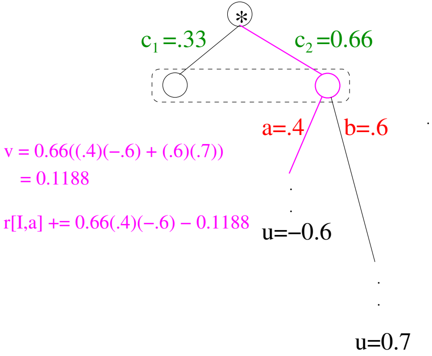

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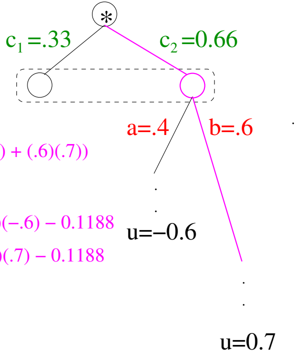
$v = 0.66((.4)(-.6) + (.6)(.7))$
$= 0.1188$

r[I,a] += 0.66(.4)(–.6) – 0.1188

r[I,b] += 0.66(.6)(.7) – 0.1188



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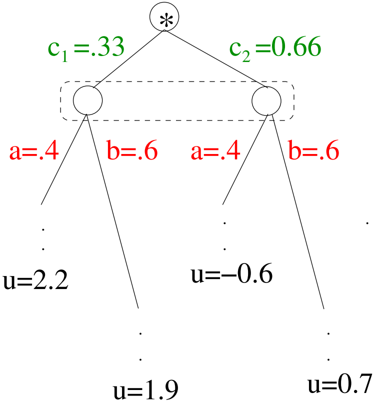
2. $\sigma_i^{t+1}(I) \leftarrow \text{RegretMatching}(r[I])$



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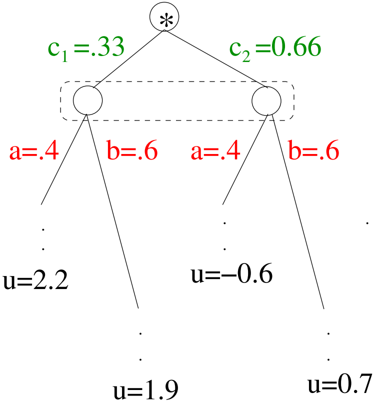
3. Update average profile $\bar{\sigma}$



CFR Extension Outline

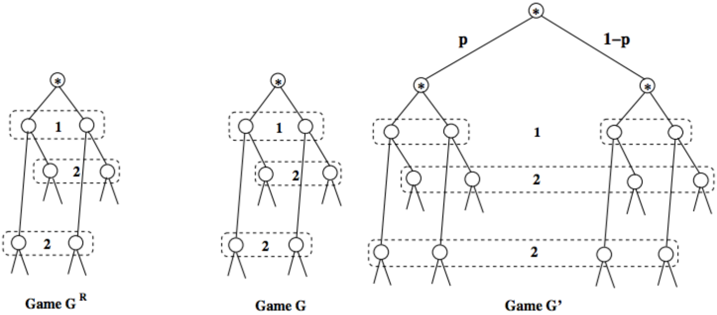
Many (20+ !) follow-up papers on CFR.

I will cover a subset:

- Restricted Nash Responses
- Monte Carlo CFR
- Imperfect Recall Abstraction
- Multiplayer and non-zero-sum
- Sequence-Form Replicator Dynamics
- CFR-BR
- CFR+

Restricted Nash Responses

Refs: [Johanson and Bowling '08, '09][Ponsen et al. '12]



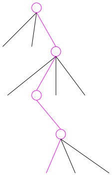
- Game $\mathbf{G}$ is some game.
- $\mathbf{G}^{\mathbf{R}}$ is a *restricted copy* (e.g. player $-i$ plays σ_{fixed})
- Nash$_i(\mathbf{G}')$ $\Leftrightarrow$ best trade-off between Nash$_i(\mathbf{G})$ and BR$_i(\sigma_{fixed})$

Monte Carlo CFR

Refs: [Lanctot et. al '09], [Gibson et al. '12], [Johanson et al. '12], [Burch et al. '12]

Sample parts of the tree: **sampled counterfactual values** $\tilde{v}_i(\sigma, I)$.

Unbiased estimator: $\mathbb{E}[\tilde{v}_i(\sigma, I)] = v_i(\sigma, I)$.



Outcome sampling

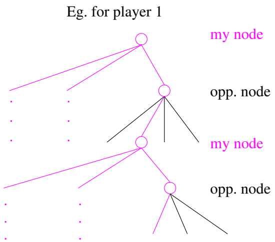
Theorem: with probability $1-p$, δ is i's min prob sampling z

$$R_i^T/T \leq \left(M_i(\sigma_i^*) \sqrt{|\max_I A(I)|} + \frac{\sqrt{2|\mathcal{I}_i||\mathcal{B}_i|}}{\sqrt{p}} \right) \left(\frac{1}{\delta} \right) \left(\frac{\Delta_{u,i}}{\sqrt{T}} \right)$$

E.g. chance sampling $\rightarrow$ sample only chance outcomes

Monte Carlo CFR: External Sampling

Refs: [Lanctot et. al ’09][Gibson ’14]



Theorem: with prob $1-p$:

$$R_i^T/T \leq \left(M_i(\sigma_i^*)\sqrt{|\max_I A(I)|} + \frac{\sqrt{2|\mathcal{I}_i||\mathcal{B}_i|}}{\sqrt{p}} \right) \left(\frac{\Delta_{u,i}}{\sqrt{T}} \right)$$

- Has worked well in (>2)-player and large action spaces
- Tartanian7, 2014 winner of 2P NL, used variant of ext. sampling

Monte Carlo CFR: Public Chance Sampling

Refs: [Johanson et. al '11][Jackson '12]

Sample only *public* chance events!

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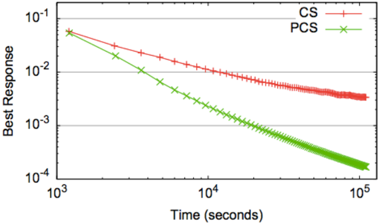

Liar's Dice (2,2)

Slumbot, 2012 winner of HULHE, used PCS

Generalized Monte Carlo CFR

Refs: [Gibson et al. '12]

Given *any* estimator for counterfactual values $\hat{v}(I, a)$ with bounded range $\hat{\Delta}$:

Theorem: with prob $1 - p$,

$$R_i^T / T \leq |\mathcal{I}_i| \left(\frac{\hat{\Delta}_i \sqrt{\max_I |A(I)|}}{\sqrt{T}} + \sqrt{\frac{\mathbf{Var}}{pT} + \frac{\mathbf{Cov}}{p} + \frac{\mathbf{E}^2}{p}} \right),$$

where:

- $\mathbf{Var}$ is max variance of diff in regret and est. regret at t,
- $\mathbf{Cov}$ is max covariance of diff in regret and est. regret at t, t',
- $\mathbf{E}$ is the max expectation of diff in regret and est. regret (bias) at t,

over all time steps t (and t'), info sets I, actions $a \in A(I)$.

Imperfect Recall Abstraction

Refs: [Waugh et al. '09][Lanctot et al. '12][Kroer & Sandholm '14, '16]

History $h \in \breve{I}$, define $X_i(h) = (\breve{I}_1, a_1), (\breve{I}_2, a_2), \cdots$ as player i's choice sequence for all I_k belonging to i in h.

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Benefits:

1. Huge savings in memory
2. Often clear what should be forgotten
3. CFR algorithm still runs(!)
   - But does it still work/converge?
   - In theory: yes! Under some (somewhat restrictive) assumptions.
   - In practice: yes, very well!

Multi (> 2) player and non-zero sum

Refs: [Abou Risk & Szafron '10][Gibson & Szafron '11][Gibson et al. '13][Gibson '14]

Generally not much known about CFR in this case.

But here again, algorithm is still well-defined.

Gibson 2014:

- Regret min. removes iteratively strictly-dominated strategies.
- Extend to *dominated actions* and counterfactual values.
- CFR removes iterative strictly-dominated actions.
- 2-player game: If $R_i^T/T < \epsilon$, converges to $2(\epsilon + \delta_u)$-Nash.

Hyperborean: winner of 2012, 2013, and 2014 3-player competitions.

Sequence-Form Replicator Dynamics

Refs: [Gatti et al. '13][Lanctot '14]

Recall Q set of choices (I, a), and $x_i(q)$ realization weight on q:

For player i, for each $q \in Q_i$, update:

$$x_i(q, t+1) = x_i(q, t)\frac{u_i(\mathbf{x}_{i \to g_q})}{u_i(\mathbf{x})}$$

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$$g_q(\mathbf{x}_i, q') = \begin{cases} 1 & \text{if } q' \in X_i(q), \\ \frac{x_i(q')}{\text{Ancestor}(q, q')} & \text{if } X_i(q) \sqsubseteq X_i(q'), \\ 0 & \text{otherwise,} \end{cases}$$

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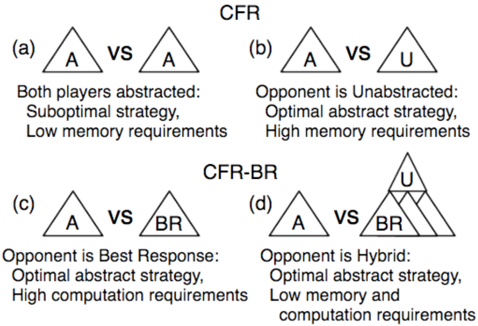
- $g_q(\mathbf{x}_i)$ is a "projection": i plays q if possible, else plays $\mathbf{x}_i$
- Implements a form of counteractual regret minimization
- In 3-player Kuhn poker, finds "best" equilibrium!

CFR-BR

Refs: [Johanson et al. ’11][Johanson et al. ’12]



- Minimize regret against a best responder
- Best responder uses *full unabstracted space*
- Use accelerated algorithms for computing best response
- Used in diabetes patient simulation [Chen & Bowling ’12]

CFR+

Refs: [Tammelin et al. '11]

Regret matching plus (RM$^+$): never accumulate negative regret!

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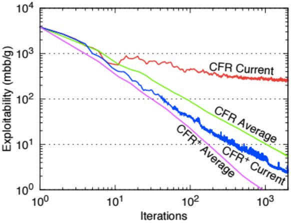
Tracking regret [Herbster & Warmuth '98]: hindsight strategy can change $(k-1)$ times.

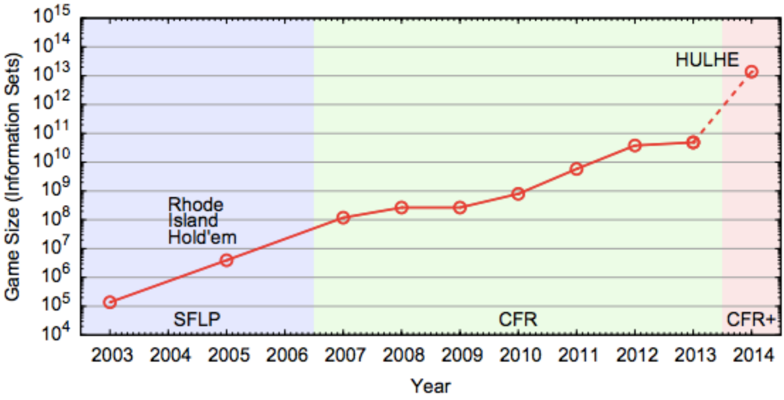
Theorem 2: T step: RM$^+$ has tracking regret $k\Delta_u \sqrt{|A|T}$.

Theorem 3: T step: CFR$^+$ has regret $O(|\mathcal{I}_1| + |\mathcal{I}_2|)\sqrt{|A|T}$.



Solving 2-player HULHE

Refs: [Bowling et al. '15]



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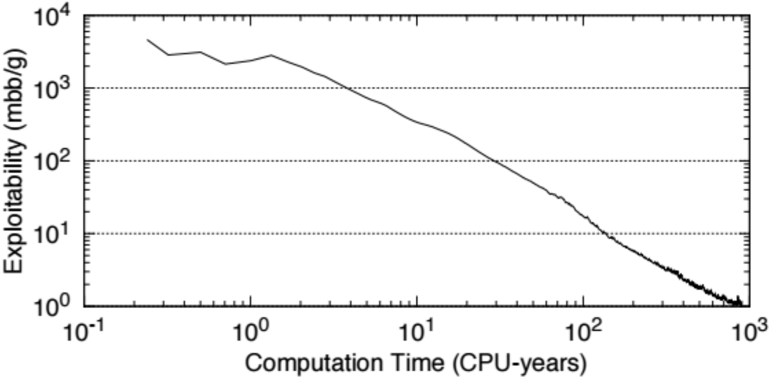

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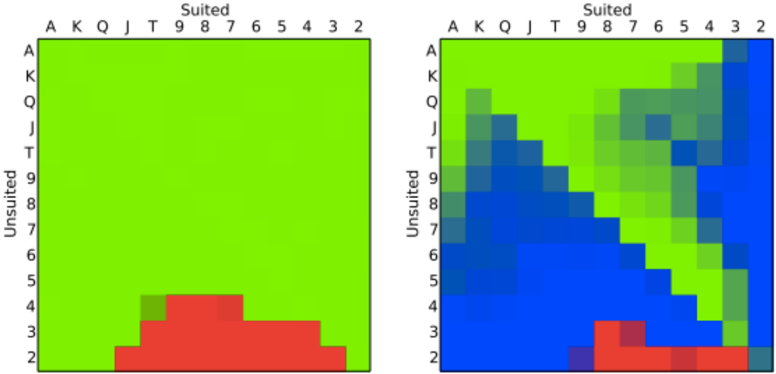

(a) first action as the dealer

(b) first action as the non-dealer after a dealer raise

Legend: fold raise call

Open Problem #1: Stronger-than-Nash?

Can a new variant of CFR converge to a:

- Sequential equilibrium?
- Trembling-hand perfect equilibrium?
- Strong equilibrium?

Open Problem #2: Correlated Equilibrium?

Does/can CFR converge to an (extensive-form) correlated equilibrium?

Other work

- FSICFR (chance-sampling variant) [Neller & Hnath '11]
- CFR with decomposition [Burch et al. '12][Jackson '14]
- Regret transfer [Brown and Sandholm '14]
- Regret-based Pruning [Brown and Sandholm '14]
- Automated abstraction and solving [Brown and Sandholm '15]
- Warm starting CFR [Brown and Sandholm '16]
- Online search [Lisý, Lanctot, and Bowling '15][Heinrich & Silver '15]
- Relationship to optimization [Waugh and Bagnell '15]
- Fictitious Self-play [Heinrich, Lanctot, and Silver '15]
- End-to-end learning [Waugh et al. '15][Heinrich and Silver '16]
- Application to security domains [Lisy, Davis, and Bowling '16]
- ...

Thanks, Questions, Info

Thank you for listening! Any questions?

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